Transfer Functions

- Convenient representation of a *linear*, dynamic model.
- A transfer function (TF) relates *one* input and *one* output:

$$\begin{matrix} x(t) \\ X(s) \end{matrix} \rightarrow \boxed{\text{system}} \rightarrow \begin{matrix} y(t) \\ Y(s) \end{matrix}$$

The following terminology is used:

$\underline{x}$	$\underline{y}$
input	output
forcing function	response
“cause”	“effect”

Definition of the transfer function:

Let $G(s)$ denote the transfer function between an input, x, and an output, y. Then, by definition

$$G(s) \triangleq \frac{Y(s)}{X(s)}$$

where:

$$Y(s) \triangleq \mathfrak{L}\left[y(t)\right]$$
$$X(s) \triangleq \mathfrak{L}\left[x(t)\right]$$

Development of Transfer Functions

<u>Example: Stirred Tank Heating System</u>

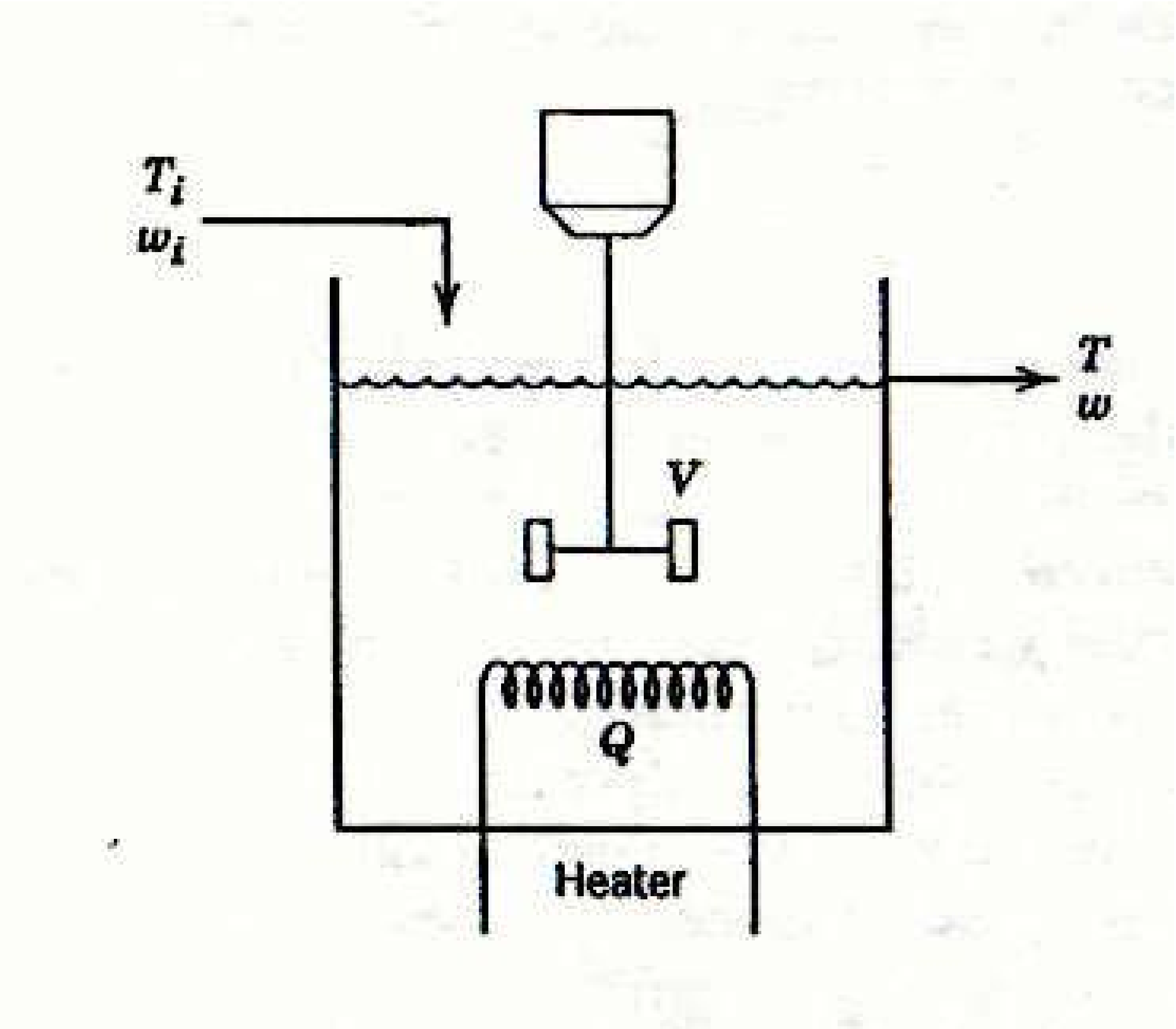


Figure 2.3 Stirred-tank heating process with constant holdup, *V*.

Recall the previous dynamic model, assuming constant liquid holdup and flow rates:

$$V\rho C\frac{dT}{dt}=wC\left(T_i-T\right)+Q \qquad (1)$$

Suppose the process is initially at steady state:

$$T\left(0\right)=\bar{T},\ T_i\left(0\right)=\bar{T}_i,\ Q\left(0\right)=\bar{Q} \qquad (2)$$

where $\bar{T} \triangleq$ steady-state value of *T*, etc. For steady-state conditions:

$$0=wC\left(\bar{T}_i-\bar{T}\right)+\bar{Q} \qquad (3)$$

Subtract (3) from (1):

$$V\rho C\frac{dT}{dt}=wC\left[\left(T_i-\bar{T}_i\right)-\left(T-\bar{T}\right)\right]+\left(Q-\bar{Q}\right) \qquad (4)$$

But,

$$\frac{dT}{dt} = \frac{d\left(T - \overline{T}\right)}{dt} \quad \text{because } \overline{T} \text{ is a constant} \qquad (5)$$

Thus we can substitute into (4-2) to get,

$$V\rho C \frac{dT'}{dt} = wC\left(T_i' - T'\right) + Q' \qquad (6)$$

where we have introduced the following "*deviation variables*", also called "perturbation variables":

$$T' \triangleq T - \overline{T}, \quad T_i' \triangleq T_i - \overline{T}_i, \quad Q' \triangleq Q - \overline{Q} \qquad (7)$$

Take $\mathfrak{L}$ of (6):

$$V\rho C\left[sT'(s) - T'(t=0)\right] = wC\left[T_i'(s) - T'(s)\right] - Q'(s) \qquad (8)$$

Evaluate $T'(t=0)$.

By definition, $T' \triangleq T - \bar{T}$. Thus at time, $t = 0$,

$$T'(0) = T(0) - \bar{T} \qquad (9)$$

But since our assumed initial condition was that the process was initially at steady state, i.e., $T(0) = \bar{T}$ it follows from (9) that $T'(0) = 0$.

Note: The advantage of using deviation variables is that the initial condition term becomes zero. This simplifies the later analysis.

Rearrange (8) to solve for $T'(s)$:

$$T'(s) = \left(\frac{K}{\tau s + 1}\right) Q'(s) + \left(\frac{1}{\tau s + 1}\right) T_i'(s) \qquad (10)$$

where two new symbols are defined:

$$K \triangleq \frac{1}{wC} \quad \text{and} \quad \tau \triangleq \frac{V\rho}{w} \tag{11}$$

Transfer Function Between Q' and T'

Suppose T_i is constant at the steady-state value. Then, $T_i(t) = \bar{T}_i \Rightarrow T_i'(t) = 0 \Rightarrow T_i'(s) = 0.$ Then we can substitute into (10) and rearrange to get the desired TF:

$$\boxed{\frac{T'(s)}{Q'(s)} = \frac{K}{\tau s + 1}} \tag{12}$$

Transfer Function Between T' and T_i':

Suppose that Q is constant at its steady-state value:

$$Q(t) = \bar{Q} \Rightarrow Q'(t) = 0 \Rightarrow Q'(s) = 0$$

Thus, rearranging

$$\boxed{\frac{T'(s)}{T_i'(s)} = \frac{1}{\tau s + 1}} \qquad (13)$$

Comments:

1. The TFs in (12) and (13) show the *individual* effects of Q and T_i on T. What about *simultaneous* changes in both Q and T_i?

- Answer: See (10). The same TFs are valid for simultaneous changes.

- Note that (10) shows that the effects of changes in both Q and T_i are *additive*. This always occurs for linear, dynamic models (like TFs) because the Principle of Superposition is valid.

2. The TF model enables us to determine the output response to any change in an input.
3. Use deviation variables to eliminate initial conditions for TF models.

Properties of Transfer Function Models

1. Steady-State Gain

The steady-state of a TF can be used to calculate the steady-state change in an output due to a steady-state change in the input. For example, suppose we know two steady states for an input, *u*, and an output, *y*. Then we can calculate the steady-state gain, *K*, from:

$$K = \frac{\bar{y}_2 - \bar{y}_1}{\bar{u}_2 - \bar{u}_1} \qquad (4\text{-}38)$$

For a linear system, *K* is a constant. But for a nonlinear system, *K* will depend on the operating condition $(\bar{u}, \bar{y})$.

Calculation of *K* from the TF Model:

If a TF model has a steady-state gain, then:

$$\boxed{K = \lim_{s \to 0} G(s)} \tag{14}$$

- This important result is a consequence of the Final Value Theorem
- *Note*: Some TF models do *not* have a steady-state gain (e.g., integrating process in Ch. 5)

2. Order of a TF Model

Consider a general n-th order, linear ODE:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{dy^{n-1}}{dt^{n-1}} + \ldots a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} +$$

$$b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \ldots + b_1 \frac{du}{dt} + b_0 u \qquad (4\text{-}39)$$

Take $\mathfrak{L}$, assuming the initial conditions are all zero. Rearranging gives the TF:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i} \qquad (4\text{-}40)$$

Definition:

The order of the TF is defined to be the order of the denominator polynomial.

Note: The order of the TF is equal to the order of the ODE.

Physical Realizability:

For any physical system, $n \geq m$ in (4-38). Otherwise, the system response to a step input will be an impulse. This can't happen.

Example:

$$a_0 y = b_1 \frac{du}{dt} + b_0 u \quad \text{and step change in } u \qquad (4\text{-}41)$$

3. Additive Property

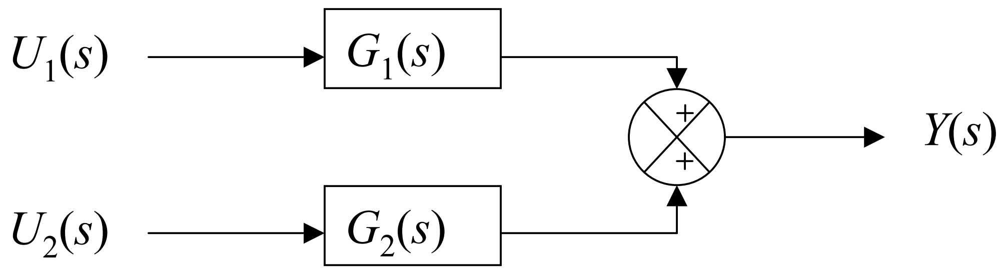
Suppose that an output is influenced by two inputs and that the transfer functions are known:

$$\frac{Y(s)}{U_1(s)} = G_1(s) \quad \text{and} \quad \frac{Y(s)}{U_2(s)} = G_2(s)$$

Then the response to changes in both U_1 and U_2 can be written as:

$$Y(s) = G_1(s)U_1(s) + G_2(s)U_2(s)$$

The graphical representation (or *block diagram*) is:



4. Multiplicative Property

Suppose that,

$$\frac{Y(s)}{U_2(s)} = G_2(s) \text{ and } \frac{U_2(s)}{U_3(s)} = G_3(s)$$

Then,

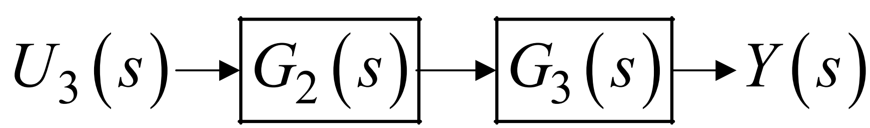
$$Y(s) = G_2(s)U_2(s) \;\; and \;\; U_2(s) = G_3(s)U_3(s)$$

Substitute,

$$Y(s) = G_2(s)G_3(s)U_3(s)$$

Or,

$$\frac{Y(s)}{U_3(s)} = G_2(s)G_3(s)$$



Linearization of Nonlinear Models

- So far, we have emphasized linear models which can be transformed into TF models.
- But most physical processes and physical models are nonlinear.
  - But over a small range of operating conditions, the behavior may be approximately linear.
  - *Conclude*: Linear approximations can be useful, especially for purpose of analysis.
- Approximate linear models can be obtained analytically by a method called "linearization". It is based on a Taylor Series Expansion of a nonlinear function about a specified operating point.

- Consider a nonlinear, dynamic model relating two process variables, u and y:

$$\frac{dy}{dt} = f\left(y,u\right) \qquad (4\text{-}60)$$

Perform a Taylor Series Expansion about $u = \bar{u}$ and $y = \bar{y}$ and truncate after the first order terms,

$$f\left(u,y\right) = f\left(\bar{u},\bar{y}\right) + \left.\frac{\partial f}{\partial u}\right|_{\bar{y}} u' + \left.\frac{\partial f}{\partial y}\right|_{\bar{y}} y' \qquad (4\text{-}61)$$

where $u' = u - \bar{u}$ and $y' = y - \bar{y}$. Note that the partial derivative terms are actually constants because they have been evaluated at the nominal operating point, $\left(\bar{u},\bar{y}\right)$.

Substitute (4-61) into (4-60) gives:

$$\frac{dy}{dt} = f\left(\bar{u},\bar{y}\right) + \left.\frac{\partial f}{\partial u}\right|_{\bar{y}} u' + \left.\frac{\partial f}{\partial y}\right|_{\bar{y}} y'$$

The steady-state version of (4-60) is:

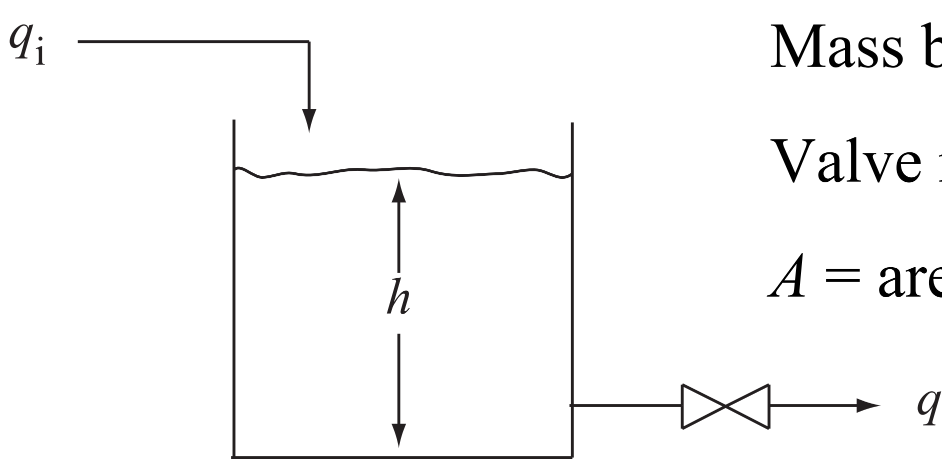
$$0 = f(\bar{u}, \bar{y})$$

Substitute into (7) and recall that $\dfrac{dy}{dt} = \dfrac{dy'}{dt}$,

$$\boxed{\frac{dy'}{dt} = \left.\frac{\partial f}{\partial u}\right|_{\bar{y}} u' + \left.\frac{\partial f}{\partial y}\right|_{\bar{y}} y'} \qquad (4\text{-}62)$$

← Linearized model

Example: Liquid Storage System



Mass balance: $A\dfrac{dh}{dt} = q_i - q$ (1)

Valve relation: $q = C_v\sqrt{h}$ (2)

A = area, C_v = constant

Chapter 4

Combine (1) and (2),

$$A\frac{dh}{dt} = q_i - C_v\sqrt{h} \tag{3}$$

Linearize $\sqrt{\ }$ term,

$$\sqrt{h} \approx \sqrt{\bar{h}} - \frac{1}{2\sqrt{\bar{h}}}\left(h - \bar{h}\right) \tag{4}$$

Or

$$\sqrt{h} \approx \sqrt{\bar{h}} - \frac{1}{R}h' \tag{5}$$

where:

$$R \triangleq 2\sqrt{\bar{h}}$$

$$h' \triangleq h - \bar{h}$$

Substitute linearized expression (5) into (3):

$$A\frac{dh}{dt} = q_i - C_v\left(\sqrt{\bar{h}} - \frac{1}{R}h'\right) \qquad (6)$$

The steady-state version of (3) is:

$$0 = \bar{q}_i - C_v\sqrt{\bar{h}} \qquad (7)$$

Subtract (7) from (6) and let $q_i' \triangleq q_i - \bar{q}_i$, noting that $\frac{dh}{dt} = \frac{dh'}{dt}$ gives the linearized model:

$$A\frac{dh'}{dt} = q_i' - \frac{1}{R}h' \qquad (8)$$

Summary:

In order to linearize a nonlinear, dynamic model:

1. Perform a Taylor Series Expansion of each nonlinear term and truncate after the first-order terms.
2. Subtract the steady-state version of the equation.
3. Introduce deviation variables.

State-Space Models

- Dynamic models derived from physical principles typically consist of one or more ordinary differential equations (ODEs).
- In this section, we consider a general class of ODE models referred to as *state-space models*.

Consider standard form for a *linear state-space model*,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{E}\boldsymbol{d} \qquad (4\text{-}90)$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} \qquad (4\text{-}91)$$

where:

$\boldsymbol{x}$ = the *state vector*

$\boldsymbol{u}$ = the control *vector* of manipulated variables (also called *control variables*)

$\boldsymbol{d}$ = the disturbance vector

$\boldsymbol{y}$ = the *output vector* of measured variables. (We use boldface symbols to denote vector and matrices, and plain text to represent scalars.)

- The elements of $\boldsymbol{x}$ are referred to as *state variables*.
- The elements of $\boldsymbol{y}$ are typically a subset of $\boldsymbol{x}$, namely, the state variables that are measured. In general, $\boldsymbol{x}$, $\boldsymbol{u}$, $\boldsymbol{d}$, and $\boldsymbol{y}$ are functions of time.
- The time derivative of $\boldsymbol{x}$ is denoted by $\dot{\boldsymbol{x}}\left(= \mathrm{d}\boldsymbol{x}/\mathrm{d}t\right)$.
- Matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{E}$ are constant matrices.

Example 4.9

Show that the linearized CSTR model of Example 4.8 can be written in the state-space form of Eqs. 4-90 and 4-91. Derive state-space models for two cases:

(a) Both c_A and T are measured.

(b) Only T is measured.

Solution

The linearized CSTR model in Eqs. 4-84 and 4-85 can be written in vector-matrix form:

$$\begin{bmatrix} \frac{dc'_A}{dt} \\ \frac{dT'}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c'_A \\ T' \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} T'_s \qquad (4\text{-}92)$$

Let $x_1 \triangleq c'_A$ and $x_2 \triangleq T'$, and denote their time derivatives by $\dot{x}_1$ and $\dot{x}_2$. Suppose that the steam temperature T_s can be manipulated. For this situation, there is a scalar control variable, $u \triangleq T'_s$, and no modeled disturbance. Substituting these definitions into (4-92) gives,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ b_2 \end{bmatrix}}_{\boldsymbol{B}} u \qquad (4\text{-}93)$$

which is in the form of Eq. 4-90 with $\boldsymbol{x} = col\ [x_1, x_2]$. (The symbol "*col*" denotes a column vector.)

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a) If both T and c_A are measured, then $\boldsymbol{y} = \boldsymbol{x}$, and $\boldsymbol{C} = \boldsymbol{I}$ in Eq. 4-91, where $\boldsymbol{I}$ denotes the 2x2 identity matrix. $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined in (4-93).

b) When only T is measured, output vector $\boldsymbol{y}$ is a scalar, $y = T'$ and $\boldsymbol{C}$ is a row vector, $\boldsymbol{C} = [0,1]$.

Note that the state-space model for Example 4.9 has $\boldsymbol{d} = \boldsymbol{0}$ because disturbance variables were not included in (4-92). By contrast, suppose that the feed composition and feed temperature are considered to be disturbance variables in the original nonlinear CSTR model in Eqs. 2-60 and 2-64. Then the linearized model would include two additional deviation variables, c'_{Ai} and T'_i.